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# Quantum field-theoretical methods in radiative transfer: unified formulae for absorption and scattering

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**Abstract.** A conceptual problem concerning the distinction between absorption and scattering is clarified. Some difficulties that have arisen in the literature are due to the neglect of interference terms between different radiation processes. The usual way of computing opacities as a sum of contributions due to scattering, free-free-absorption, etc, is not strictly valid. Accurate calculation of spectral lineshapes is possible with two general, but apparently not widely known, formulae given in this paper. These formulae are evaluated, in the dipole approximation, for the special case of cyclotron absorption and scattering in uniform magnetic fields.

## 1. Introduction

Resonant scattering of light is a well known phenomenon playing an important role in atomic spectroscopy, plasma physics and astrophysics. Stimulated by the discovery of cyclotron lines in x-ray pulsars (Trümper *et al* 1978) the analogous process involving cyclotron radiation in strong magnetic fields has recently attracted much interest (see Mészáros 1984 and references therein). Radiation processes in magnetic fields merit a careful theoretical investigation because many of the complexities of atomic physics can be sidestepped, and the fundamental problem brought into a sharper focus. On the other hand, a uniform magnetic field makes the problem just complex enough so that the underlying basic physics is not obscured by some popular but deceptive simplifications (like, e.g., the dipole approximation or the use of polarisation averages).

Here we encounter the following curious problem. While the usual Feynman rules give us a well defined scattering cross section due to electrons with a particular speed, the total opacity is formally infinite, if one integrates over the electron velocity distribution. This is, of course, due to the appearance of a 'resonant denominator', and the usual remedy is to add an imaginary part, $i\gamma$, corresponding to the finite lifetime of the intermediate state. While this recipe provides a very useful approximation, it also creates several new problems. One of them is the question as to what choice for $\gamma$ gives the most accurate results. At least two different prescriptions can be found in the literature. Ventura (1979) uses a damping proportional to the square of the photon frequency, whereas, e.g., Kirk and Mészáros (1980) use a $\gamma$ that is independent of frequency. The difference is important only in the far wings of the cyclotron line, where Ventura's choice appears to be more accurate, whereas the

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alternative is perhaps more plausible, because it is the inverse lifetime of the excited state.

A more serious problem is the fact that introducing a damping term blurs the distinction between absorption and scattering (Melrose 1981). Although it is physically obvious that absorption followed by re-emission is equivalent to scattering, it is not clear how to do the book-keeping right. We cannot simply ignore cyclotron processes, because we need them to compute the Landau level populations. We cannot ignore the scattering terms because they dominate over the pure cyclotron cross section in the far wings. Finally, we cannot just add cyclotron and scattering coefficients, because this would give the wrong opacity (the value in the line core would be doubled).

Of course, it is possible to circumvent this problem by using physical intuition to make various *ad hoc* changes in the kinetic equations, e.g., by dropping some terms in the rate equations (the kinetic equation for the electrons) while retaining them in the transfer equation (the kinetic equation for the photons). This is the approach adopted in most previous work on this problem. Cooper *et al* (1983) discussed a related problem in the context of atomic lines.

A natural starting point for the discussion of these issues would appear to be the set of Boltzmann equations for electrons and photons, e.g., as they were written down by Kirk and Melrose (1986). However, these equations are meaningful only if a clear distinction between absorption/scattering and other processes can be made. Unfortunately, expansion in powers of the coupling constant (which appears in $\gamma$) renders the kinetic coefficients meaningless, because they are not integrable. How can the different processes occurring in the plasma be consistently described? What is the true cyclotron lineshape, from the core into the wings? It is my aim in this paper to clarify some of these conceptual difficulties, and to show how the opacity can be computed, at least in principle.

## 2. Application of the fluctuation/dissipation theorem

Information about the interaction between radiation and a medium is contained in the dielectric tensor. The absorption coefficient can be derived from the antiHermitian part of this tensor or, equivalently, from the conductivity tensor. As is well known, the fluctuation/dissipation theorem can be used to express the conductivity tensor in terms of the current-current correlations, leading to a so-called Kubo formula (Ziman 1969). Kubo formulae were first derived for systems near thermal equilibrium (*linear* response) but it seems that a variant of the fluctuation/dissipation theorem can also be applied to systems far from equilibrium, e.g., in plasma physics when dealing with non-Maxwellian electron velocity distributions (Sitenko 1967, equation (2.70)). For the sake of completeness, I give here an outline of the derivation.

Integrating the von Neumann equation for the density matrix of the electron system, $\mathrm{d}\rho/\mathrm{d}t = (\mathrm{i}/\hbar)[\rho, H']$, one finds to first order in the perturbation $H'$:

$$\rho(t) = \rho(-\infty) + \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}t' \,[\rho(-\infty), H'(t')]. \tag{1}$$

The perturbing potential is $H'(t) = -\int \mathrm{d}^3x\, A_\mu(x, t) j_\mu(x, t)$. The current density, $j_\mu$, is related to the current operator $j_\mu^0$ in the absence of the external field by $j_\mu = j_\mu^0 - (e^2/m)NA_\mu$, where $N$ is the total electron density. Multiplying (1) with the current

density operator and taking the trace one finds

$$\langle j_\mu(t)\rangle = -\frac{e^2}{m} NA_\mu(t) - \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}t'\, \mathrm{Tr}[\rho(-\infty), j_\nu(t')] j_\mu(t) A_\nu(t') \tag{2}$$

$$= -\frac{e^2}{m} NA_\mu(t) + \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}t'\, \langle [j_\mu(t), j_\nu(t')]\rangle A_\nu(t') \tag{3}$$

using the cyclic invariance of the trace. (I have simplified the notation by not making the integration over $x'$ and the summation over $\nu$ explicit.) The dielectric tensor can then be expressed in the form

$$\varepsilon_{\mu\nu} = \delta_{\mu\nu} - \frac{\omega_\mathrm{p}^2}{(\omega+\mathrm{i}0)^2}\delta_{\mu\nu} + \frac{\mathrm{i}/\varepsilon_0\hbar}{(\omega+\mathrm{i}0)^2}\int_0^\infty \mathrm{d}t \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}(kx-\omega t)}\langle j_\mu(x,t) j_\nu(0,0) - j_\nu(0,0) j_\mu(x,t)\rangle. \tag{4}$$

Here, $\omega_\mathrm{p} = (Ne^2/\varepsilon_0 m)^{1/2}$ would be the plasma frequency if all electrons were free, and it is assumed that the medium is *statistically* homogeneous and stationary (on the microscopic scale). But no other assumption is necessary about the initial state density matrix $\rho(-\infty)$, i.e. the system need not be near thermal equilibrium. I believe that it is possible to treat also very intense light beams (laser light), since we are free to consider a strong field $A_\mu(x,t)$ as part of the 'unperturbed' system, and to study an infinitesimally small additional 'test field'. (Of course, in such a case the current correlations may be hard to calculate.)

Taking now the antiHermitian part of the dielectric tensor (for $\omega > 0$), we find the following Kubo formula for the absorption coefficient (in $\mathrm{m}^{-1}$) of a wave with polarisation vector $\boldsymbol{e}_\mu$:

$$\kappa = \frac{\mu_0 c}{2\hbar\omega}\sum_{\mu\nu} \boldsymbol{e}_\mu^* \boldsymbol{e}_\nu \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}(kx-\omega t)}\langle j_\mu(x,t) j_\nu(0,0) - j_\nu(0,0) j_\mu(x,t)\rangle. \tag{5}$$

Here we have assumed that the plasma is so tenuous that the refractive index is close to 1. Similarly, the emissivity ($\mathrm{W\,m^{-3}\,Hz^{-1}\,sr^{-1}}$) can be written

$$\varepsilon = \frac{\mu_0\omega^2}{8\pi^2 c}\sum_{\mu\nu} \boldsymbol{e}_\mu^* \boldsymbol{e}_\nu \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}(kx-\omega t)}\langle j_\nu(0,0) j_\mu(x,t)\rangle. \tag{6}$$

It is perhaps worth pointing out that stimulated emission terms are included in (5), but not in (6). These terms always occur together with absorption terms; in fact, the current commutator in (5) can be interpreted as 'absorption' minus 'stimulated emission'. The emissivity (6) includes only spontaneous emission—it would be a mistake to add stimulated emission here.

In the theory of spectral lineshapes it is customary to express the absorption coefficient as a Fourier integral of the dipole-moment autocorrelation function (van Vleck and Huber 1977, Mihalas 1978). The equations here are more general, because they take account of spatial dispersion and describe the angular and polarisation dependence properly. The divergence problems mentioned in the introduction are solved in a very natural way—they are just the result of too simple an approximation for the current-current correlations: since the unperturbed electron propagators are used, the correlations persist to infinity, whereas it is physically clear that in reality, due to interactions with photons, the correlations must eventually decay. Hence the Fourier integrals are always well defined, if physically reasonable approximations to the current-current correlations can be found.

In what follows I shall assume that the equations (5) and (6) are general, and show how the usual expressions for cyclotron absorption and emission and for scattering can be derived from them as special cases. This is to verify that these formulae can in fact be used (in a more or less straightforward way) to calculate the opacities for free electrons in a magnetic field. But this approach is not limited to free electrons; the application to atoms is just slightly more involved (see, e.g., Baym 1969, ch 13).

### 3. The quantum field-theoretical machinery

In the evaluation of the correlation functions special attention must be paid to the ordering of the operators. The 'closed timepath' formalism is useful for doing this; information can be found in several recent reviews (Chou *et al* 1985, Rammer and Smith 1986, Landsman and van Weert 1987). I use $[\![ \ldots ]\!]$ to indicate time ordering along a closed timepath extended over 'forward' times $\underline{t}$ from $-\infty$ to $+\infty$, and over 'backward' times $\bar{t}$ from $+\infty$ to $-\infty$. Denoting by $\mathbf{T}$ the usual time-ordering operator and by $\tilde{\mathbf{T}}$ its conjugate, we have

$$\begin{aligned}
&[\![ A(\underline{t})B(\underline{t}') ]\!] = \mathbf{T}A(t)B(t') \\
&[\![ A(\bar{t})B(\underline{t}') ]\!] = A(t)B(t') \\
&[\![ A(\underline{t})B(\bar{t}') ]\!] = \pm B(t')A(t) \\
&[\![ A(\bar{t})B(\bar{t}') ]\!] = \tilde{\mathbf{T}}A(t)B(t')
\end{aligned} \tag{7}$$

i.e. all backward times $\bar{t}$ are considered 'later' than all forward times $\underline{t}'$. Correlation functions can then be formally defined by, e.g.,

$$\langle j_\mu(x,t)j_\nu(0,0)\rangle = \left\langle \left[\!\!\left[ j_\mu(x,\bar{t})j_\nu(0,\underline{0}) \exp\left( -\mathrm{i}\hbar^{-1} \oint \mathrm{d}t'\, H'(t') \right) \right]\!\!\right] \right\rangle \tag{8}$$

where $H'(t')$ is the perturbing potential, $\oint \mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{d}\underline{t} - \int_{-\infty}^{\infty} \mathrm{d}\bar{t}$, and the time ordering ensures that in the expansion of the exponential all operators fall in the right place. The correlation functions in (5) and (6) above are special cases with one time on the forward and the other on the backward branch.

As usual, the current-density operator can be expressed in terms of the creation and annihilation operators:

$$\begin{aligned}
j_\mu(x,t) = \Bigg[ &\mathrm{i}\frac{e\hbar}{2m}\Psi^+(2)\nabla_{\mu 1}\Psi(1) - \mathrm{i}\frac{e\hbar}{2m}\nabla_{\mu 2}\Psi^+(2)\Psi(1) \\
&- \frac{e^2}{m}A_\mu(x,t)\Psi^+(2)\Psi(1) \Bigg]_{\substack{x_1=x,\,t_1=t \\ x_2=x,\,t_2=t}}
\end{aligned} \tag{9}$$

For simplicity we shall consider in this paper only 'electrons' without spin, and with charge $-e$ (i.e. $e>0$). The current correlations can then be calculated from

$$\begin{aligned}
&\langle [\![ j_\mu(x,\bar{t})j_\nu(0,\underline{0}) ]\!] \rangle \\
&\quad = -\frac{e^2\hbar^2}{4m^2}\Bigg[ (\nabla_{\mu 3} - \nabla_{\mu 4} + \mathrm{i}eA_{\mu 3} + \mathrm{i}eA_{\mu 4}) \\
&\qquad \times (\nabla_{\nu 1} - \nabla_{\nu 2} + \mathrm{i}eA_{\nu 1} + \mathrm{i}eA_{\nu 2}) \langle [\![ \Psi^+(\bar{4})\Psi(\bar{3})\Psi^+(\underline{2})\Psi(\underline{1}) ]\!] \rangle \Bigg]_{\substack{4\to 3- \\ 2\to 1+}}
\end{aligned} \tag{10}$$

Wick's theorem allows the decomposition into single-particle Green functions:

$$\langle[\![\Psi^\dagger(\bar{4})\Psi(\bar{3})\Psi^\dagger(\underline{2})\Psi(\underline{1})]\!]\rangle$$
$$= \langle[\![\Psi(\underline{1})\Psi^\dagger(\bar{4})]\!]\rangle\langle[\![\Psi(\bar{3})\Psi^\dagger(\underline{2})]\!]\rangle$$
$$+ \langle[\![\Psi(\underline{1})\Psi^\dagger(\underline{2})]\!]\rangle\langle[\![\Psi(\bar{3})\Psi^\dagger(\bar{4})]\!]\rangle. \tag{11}$$

The 'thermal' Green functions, $G(2,1) = -\mathrm{i}\langle[\![\Psi(2)\Psi^\dagger(1)]\!]\rangle$, contain information both about how the particles move and which states are occupied (Kadanoff and Baym 1962). For a non-relativistic non-degenerate magnetised electron plasma at temperature $T = 1/\beta$ we can write

$$G(r_2, t, r_1, 0) = \mathrm{i}\mathrm{e}^{\beta\mu}\langle r_2|\mathrm{e}^{-(\beta+\mathrm{i}t)H}|r_1\rangle - \mathrm{i}\Theta_\mathrm{o}(t)\langle r_2|\mathrm{e}^{-\mathrm{i}tH}|r_1\rangle \tag{12}$$

with a simple generalisation $\Theta_\mathrm{o}(t_2 - t_1)$ of Heaviside's step function to times on the closed timepath (Chou *et al* 1985). Here, and in (14) below, we set $\hbar = 1$. The chemical potential $\mu$ is related to the particle density by

$$N = \mathrm{e}^{\beta\mu}(2\pi)^{-3/2}\frac{m\omega_\mathrm{B}}{\hbar^2}\left(\frac{\beta}{m}\right)^{1/2}[1-\exp(-\beta\hbar\omega_\mathrm{B})]^{-1} \tag{13}$$

and the exponentials $\mathrm{e}^{-\mathrm{i}Ht}$ involve just combinations of free particle and harmonic oscillator propagators (Feynman and Hibbs 1965):

$$\langle r_2|\mathrm{e}^{-(\beta+\mathrm{i}t)H}|r_1\rangle$$
$$= (2\pi)^{-3/2}m\omega_\mathrm{B}\left(\frac{\beta+\mathrm{i}t}{m}\right)^{-1/2}\frac{1}{1-\mathrm{e}^{-\omega_\mathrm{B}(\beta+\mathrm{i}t)}}$$
$$\times\exp\left(-\frac{m(z_2-z_1)^2}{2(\beta+\mathrm{i}t)} - \frac{m\omega_\mathrm{B}}{4}[(x_2-x_1)^2+(y_2-y_1)^2]\frac{1+\exp[-\omega_\mathrm{B}(\beta+\mathrm{i}t)]}{1-\exp[-\omega_\mathrm{B}(\beta+\mathrm{i}t)]}\right)$$
$$\times\exp\left(-\mathrm{i}\frac{m\omega_\mathrm{B}}{2}(x_2+x_1)(y_2-y_1)\right). \tag{14}$$

This formula is, of course, gauge dependent; $(A_x, A_y, A_z) = (0, Bx, 0)$ was adopted. Substitution into the equations (10) and (11) above will lead to the well known non-relativistic expressions for the cyclotron absorption coefficient.

Another primary ingredient of the quantum field-theoretical technique is the photon propagator/density matrix:

$$\langle[\![A_\mu(x,t)A_\nu(0,0)]\!]\rangle$$
$$= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{p,q=1,2}\frac{\hbar}{2\varepsilon_0\omega}[\rho_{pq}\boldsymbol{u}^*_{p_\mu}\boldsymbol{u}_{q_\nu}\mathrm{e}^{-\mathrm{i}(kx-\omega t)} + \rho_{qp}\boldsymbol{u}_{p_\mu}\boldsymbol{u}^*_{q_\nu}\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$+ \delta_{pq}\boldsymbol{u}_{p_\mu}\boldsymbol{u}^*_{q_\nu}\Theta_\mathrm{o}(t)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \delta_{pq}\boldsymbol{u}^*_{p_\mu}\boldsymbol{u}_{q_\nu}\Theta_\mathrm{o}(-t)\,\mathrm{e}^{-\mathrm{i}(kx-\omega t)}]. \tag{15}$$

Here, $\rho_{pq}$ is the expectation value of $a^\dagger_p a_q$, where $a^\dagger_p$ creates a photon in the polarisation mode $p$, and $\boldsymbol{u}_p$ is the polarisation vector of that mode. Expressed in terms of the more conventional Stokes parameters, these numbers are

$$\rho_{pq} = \frac{2\pi^2c^2}{\hbar\omega^3}\begin{pmatrix} I+Q & U+\mathrm{i}V \\ U-\mathrm{i}V & I-Q \end{pmatrix} \tag{16}$$

when $I$, $Q$, $U$, $V$ are measured in $\mathrm{W\,m^{-2}\,Hz^{-1}\,sr^{-1}}$.

These expressions are given here just to give an idea of what is involved in the 'closed timepath' formalism, and to show that the formalism is able to accommodate all those complications which we are going to ignore in what follows. My main concern here is to illustrate the potential of the method.

## 4. Cyclotron absorption and emission

To simplify the following discussion we shall now disregard spatial dispersion, i.e. let $k \to 0$. If we can treat the electrons as an ideal gas we have

$$\int \mathrm{d}^3x \langle j_\mu(x, t) j_\nu(0, 0)\rangle = Ne^2 \langle v_\mu(t) v_\nu(0)\rangle \tag{17}$$

and we are left with the much simpler velocity correlations. The velocity operators can be defined as

$$mv_\mu = p_\mu + eA_\mu. \tag{18}$$

It will be convenient to use rotating coordinates, $v_\pm = (v_x \pm \mathrm{i}v_y)/\sqrt{2}$. The operators $v_\pm$ can serve as ladder operators (Canuto and Ventura 1977), i.e. $v_+$ moves an electron to the next-higher Landau level, and $v_-$ moves it down by one. They satisfy the commutation relation

$$[v_-, v_+] = \frac{\hbar\omega_B}{m} \tag{19}$$

and they allow the Hamiltonian to be written in the form

$$H = mv_+v_- + \tfrac{1}{2}mv_z^2. \tag{20}$$

If the electron gas is in thermal equilibrium (at temperature $T$), it is easy to see that

$$\langle v_+(t) v_-(0)\rangle = \langle n\rangle (\hbar\omega_B/m) \exp(\mathrm{i}\omega_B t) \tag{21}$$

$$\langle v_-(t) v_+(0)\rangle = \langle n+1\rangle (\hbar\omega_B/m) \exp(-\mathrm{i}\omega_B t) \tag{22}$$

$$\langle v_z(t) v_z(0)\rangle = T/m \tag{23}$$

where we have switched to Heisenberg operators, $v_\pm(t) = v_\pm \exp(\pm\mathrm{i}\omega_B t)$, $v_z(t) = v_z$, and have denoted the mean Landau quantum number by $\langle n\rangle = 1/[\exp(\hbar\omega_B/T) - 1]$. All other velocity correlations vanish.

The coefficient of cyclotron absorption is then

$$\kappa = \frac{\mu_0 cNe^2}{2\hbar\omega} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \sum_{\mu\nu} e_\mu^* e_\nu \langle v_\mu(t) v_\nu(0) - v_\nu(0) v_\mu(t)\rangle. \tag{24}$$

In the sum only two terms are non-zero:

$$\langle v_+(t) v_-(0) - v_-(0) v_+(t)\rangle = \frac{\hbar\omega_B}{m} [\langle n\rangle - \langle n+1\rangle] \exp(\mathrm{i}\omega_B t) \tag{25}$$

$$\langle v_-(t) v_+(0) - v_+(0) v_-(t)\rangle = \frac{\hbar\omega_B}{m} [\langle n+1\rangle - \langle n\rangle] \exp(-\mathrm{i}\omega_B t). \tag{26}$$

Since opacities are meaningful only for positive frequencies, we can ignore the first term, whose Fourier transform is proportional to $\delta(\omega + \omega_B)$. We are left with

$$\kappa = \frac{\mu_0 cNe^2\omega_B}{2m\omega} |e_-|^2 [\langle n+1\rangle - \langle n\rangle] 2\pi\delta(\omega - \omega_B)$$

$$= 4\pi^2\alpha \frac{\hbar}{m} N |e_-|^2 \delta(\omega - \omega_B). \tag{27}$$

(My convention is to transform to rotating coordinates *after* taking the complex conjugate, so that the polarisation component $e_+^*$ is in fact the complex conjugate of $e_-$.) Given all the simplifications, the formula is the best expression we can obtain for the cyclotron absorption coefficient. The first term proportional to $\langle n+1\rangle$ can be attributed to 'absorption', the second one containing $\langle n\rangle$ to 'stimulated emission'. Apart from a factor $\hbar\omega^3/4\pi^2c^2$ it is identical to the emissivity

$$\varepsilon = \alpha \frac{\hbar^2\omega^3}{mc^2} N\langle n\rangle |e_-|^2 \delta(\omega-\omega_B) \tag{28}$$

which describes 'spontaneous emission'. This is, of course, the well known relation between Einstein's $A$ and $B$ coefficients.

It is a curious fact that, when spatial dispersion is ignored ($k\to 0$), the cyclotron absorption cross section is independent of the occupation of the Landau levels. A 'population inversion' appears to be impossible, because above every Landau level there is another one with a still larger transition matrix element (proportional to $\sqrt{n+1}$). However, when the current correlations are considered, instead of the velocity correlations, a maser effect becomes indeed possible, and is of course well known in plasma physics (electromagnetic cyclotron instability).

## 5. Scattering

At frequencies different from the cyclotron frequency other processes, notably scattering, contribute to the opacity. How can this be derived from the Kubo formula (5)? The basic idea is very simple. Let the electron be in the field of a wave and calculate how the radiation perturbs the velocity fluctuations. The procedure is thus very similar to the usual prescription for calculating a scattering cross section: take an incident wave, compute the perturbed motion of the system, and find out how the perturbed system radiates. The present formalism is different, however, in the sense that we are not interested in the *amplitude* of the scattered wave, but shall calculate directly the *intensity* (more precisely, the Stokes parameters) of the scattered light.

It is here that we need the 'closed timepath' formalism described briefly in § 3. For the unperturbed velocity correlations we have

$$\begin{aligned}\langle\!\langle v_\mu(t)v_\nu(0)\rangle\!\rangle &= (\hbar\omega_B/m)\langle n\rangle[\delta_{\mu+}\delta_{\nu-}\exp(i\omega_B t)+\delta_{\mu-}\delta_{\nu+}\exp(-i\omega_B t)]\\ &+(\hbar\omega_B/m)[\Theta_o(-t)\delta_{\mu+}\delta_{\nu-}\exp(i\omega_B t)+\Theta_o(t)\delta_{\mu-}\delta_{\nu+}\exp(-i\omega_B t)]\\ &+(T/m)\delta_{\mu z}\delta_{\nu z}.\end{aligned} \tag{29}$$

The true velocities are given by

$$u_\mu(t) = v_\mu(t)+(e/m)A_\mu(t) \tag{30}$$

where $A(t)$ is the field of the wave at the position of the electron. Note that the homogeneous magnetic field is already included in $v(t)$ by virtue of the definition (18). What we need to compute, then, is

$$\begin{aligned}\langle\!\langle u_\mu(t)u_\nu(0)\rangle\!\rangle_A &= \langle\!\langle v_\mu(t)v_\nu(0)\rangle\!\rangle_A+(e/m)\langle\!\langle A_\mu(t)v_\nu(0)\rangle\!\rangle_A\\ &+(e/m)\langle\!\langle v_\mu(t)A_\nu(0)\rangle\!\rangle_A+(e^2/m^2)\langle\!\langle A_\mu(t)A_\nu(0)\rangle\!\rangle_A.\end{aligned} \tag{31}$$

The perturbing radiation field also appears implicitly in the expansion of

$$\langle\!\langle v_\mu(t)v_\nu(0)\rangle\!\rangle_A = \left\langle\!\!\left[\!\!\left[ v_\mu(t)v_\nu(0)\exp\left(-\frac{i}{\hbar}\oint dt'\,H'(t')\right)\right]\!\!\right]\!\!\right\rangle \tag{32}$$

with $H'(t') = eA(t')v(t')$. To reproduce the usual expressions for the scattering coefficient it is sufficient to expand up to second order in $A$:

$$\langle\!\langle v_\mu(t)v_\nu(0)\rangle\!\rangle_A = \langle\!\langle v_\mu(t)v_\nu(0)\rangle\!\rangle - \frac{ie}{\hbar}\oint dt_1 \langle\!\langle v_\mu(t)v_\nu(0)v_\alpha(t_1)A_\alpha(t_1)\rangle\!\rangle$$
$$-\frac{e^2}{2\hbar^2}\oint dt_1 \oint dt_2 \langle\!\langle v_\mu(t)v_\nu(0)v_\alpha(t_1)A_\alpha(t_1)v_\beta(t_2)A_\beta(t_2)\rangle\!\rangle. \tag{33}$$

Applying Wick's theorem, and keeping only terms quadratic in $A$, one obtains

$$\langle\!\langle u_\mu(t)u_\nu(0)\rangle\!\rangle_A \simeq \frac{e^2}{m^2}\langle\!\langle A_\mu(t)A_\nu(0)\rangle\!\rangle$$
$$-i\frac{e^2}{\hbar m}\oint dt_1 \langle\!\langle v_\mu(t)v_\alpha(t_1)\rangle\!\rangle\langle\!\langle A_\alpha(t_1)A_\nu(0)\rangle\!\rangle$$
$$-i\frac{e^2}{\hbar m}\oint dt_1 \langle\!\langle A_\mu(t)A_\alpha(t_1)\rangle\!\rangle\langle\!\langle v_\alpha(t_1)v_\nu(0)\rangle\!\rangle$$
$$-\frac{e^2}{\hbar^2}\oint dt_1 \oint dt_2 \langle\!\langle v_\mu(t)v_\alpha(t_1)\rangle\!\rangle\langle\!\langle A_\alpha(t_1)A_\beta(t_2)\rangle\!\rangle\langle\!\langle v_\beta(t_2)v_\nu(0)\rangle\!\rangle \tag{34}$$

where the last term is the sum of two identical terms resulting from the application of Wick's theorem to the last term in (33).

In the following, we shall consider cold plasma, i.e $T = 0$, $\langle n\rangle = 0$, and retain only terms proportional to the incident flux, $F_0$ (in W m$^{-2}$). Then we are left with

$$\langle\!\langle v_\mu(t)v_\nu(0)\rangle\!\rangle = \frac{\hbar\omega_B}{m}[\delta_{\mu-}\delta_{\nu+}\Theta_0(t)\exp(-i\omega_B t) + \delta_{\mu+}\delta_{\nu-}\Theta_0(-t)\exp(i\omega_B t)] \tag{35}$$

and

$$\langle\!\langle A_\mu(t)A_\nu(0)\rangle\!\rangle = \frac{\mu_0 c}{2\omega^2}F_0[e_\mu e_\nu^* e^{-i\omega t} + e_\mu^* e_\nu e^{i\omega t}] \tag{36}$$

if the incident wave is completely polarised and has the polarisation vector $e_\mu$. The convolution integrals in (34) are easily performed using the identities

$$\oint dt' \exp[-i\omega_2(t_2 - t')]\exp[-i\omega_1(t' - t_1)] = 0 \tag{37}$$

$$\oint dt' \exp[-i\omega_2(t_2 - t')]\Theta_0(t' - t_1)\exp[-i\omega_1(t' - t_1)] = \frac{i}{\omega_2 - \omega_1 - i0}\exp[-i\omega_2(t_2 - t_1)] \tag{38}$$

$$\oint dt' \Theta_0(t_2 - t')\exp[-i\omega_2(t_2 - t')]\exp[-i\omega_1(t' - t_1)] = \frac{i}{\omega_1 - \omega_2 + i0}\exp[-i\omega_1(t_2 - t_1)] \tag{39}$$

which are valid for $t_1$, $t_2$ both on the forward or backward time branches. One then finds

$$\oint dt_1 \langle\!\langle v_\mu(t)v_\alpha(t_1)\rangle\!\rangle\langle\!\langle A_\alpha(t_1)A_\nu(0)\rangle\!\rangle$$
$$= -i\frac{\mu_0 c\hbar}{2m\omega^2}F_0\left[\left(\delta_{\mu-}e_-\frac{\omega_B}{\omega_B - \omega - i0} + \delta_{\mu+}e_+\frac{\omega_B}{\omega_B + \omega + i0}\right)e_\nu^* e^{-i\omega t}\right.$$
$$\left. + \left(\delta_{\mu-}e_-^*\frac{\omega_B}{\omega_B + \omega - i0} + \delta_{\mu+}e_+^*\frac{\omega_B}{\omega_B + \omega + i0}\right)e_\nu e^{i\omega t}\right] \tag{40}$$

$$\oint \mathrm{d}t_1 \langle [\![ A_\mu(t) A_\alpha(t_1) ]\!] \rangle \langle [\![ v_\alpha(t_1) v_\nu(0) ]\!] \rangle$$

$$= -\mathrm{i} \frac{\mu_0 c \hbar}{2m\omega^2} F_0 \Bigg[ \left( \delta_{\nu+} \boldsymbol{e}_+^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega + \mathrm{i}0} + \delta_{\nu-} \boldsymbol{e}_-^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega - \mathrm{i}0} \right) \boldsymbol{e}_\mu \, \mathrm{e}^{-\mathrm{i}\omega t}$$

$$+ \left( \delta_{\nu+} \boldsymbol{e}_+ \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega + \mathrm{i}0} + \delta_{\nu-} \boldsymbol{e}_- \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega - \mathrm{i}0} \right) \boldsymbol{e}_\mu^* \, \mathrm{e}^{\mathrm{i}\omega t} \Bigg] \tag{41}$$

and

$$\oint \mathrm{d}t_1 \oint \mathrm{d}t_2 \langle [\![ v_\mu(t) v_\alpha(t_1) ]\!] \rangle \langle [\![ A_\alpha(t_1) A_\beta(t_2) ]\!] \rangle \langle [\![ v_\beta(t_2) v_\nu(0) ]\!] \rangle$$

$$= -i \frac{\mu_0 c \hbar^2}{2m^2\omega^2} F_0 \Bigg[ \left( \delta_{\mu-} \boldsymbol{e}_- \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega - \mathrm{i}0} + \delta_{\mu+} \boldsymbol{e}_+ \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega + \mathrm{i}0} \right)$$

$$\times \left( \delta_{\nu+} \boldsymbol{e}_+^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega + \mathrm{i}0} + \delta_{\nu-} \boldsymbol{e}_-^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega - \mathrm{i}0} \right) \mathrm{e}^{-\mathrm{i}\omega t}$$

$$+ \left( \delta_{\mu-} \boldsymbol{e}_-^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega - \mathrm{i}0} + \delta_{\mu+} \boldsymbol{e}_+^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega + \mathrm{i}0} \right)$$

$$\times \left( \delta_{\nu+} \boldsymbol{e}_+ \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega + \mathrm{i}0} + \delta_{\nu-} \boldsymbol{e}_- \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega - \mathrm{i}0} \right) \mathrm{e}^{\mathrm{i}\omega t} \Bigg]. \tag{42}$$

Combining these results into (34) one finds, after a little algebra, the velocity fluctuations

$$\langle [\![ u_\mu(t) u_\nu(0) ]\!] \rangle_A = \frac{2\pi\alpha\hbar}{m^2\omega^2} F_0 \left( \delta_{\mu+} \boldsymbol{e}_+ \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega + \mathrm{i}0} + \delta_{\mu-} \boldsymbol{e}_- \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega - \mathrm{i}0} - \boldsymbol{e}_\mu \right)$$

$$\times \left( \delta_{\nu+} \boldsymbol{e}_+^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega + \mathrm{i}0} + \delta_{\nu-} \boldsymbol{e}_-^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega - \mathrm{i}0} - \boldsymbol{e}_\nu^* \right) \mathrm{e}^{-\mathrm{i}\omega t} \tag{43}$$

where, anticipating that we need to calculate the emissivity for positive frequencies, we have retained only the terms containing $\mathrm{e}^{-\mathrm{i}\omega t}$.

Applying now the Kubo formula (6) we find for the scattered light:

$$\varepsilon(\omega') = \frac{\alpha\hbar\omega'^2}{2\pi c^2} N \sum_{\mu,\nu} \boldsymbol{e}_\mu'^* \boldsymbol{e}_\nu' \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}\omega' t} \langle [\![ u_\mu(t) u_\nu(0) ]\!] \rangle_A \tag{44}$$

$$= \frac{\alpha^2\hbar^2}{m^2c^2} N F_0 \sum_{\mu,\nu} \boldsymbol{e}_\mu'^* \boldsymbol{e}_\nu' \left( \delta_{\mu+} \boldsymbol{e}_+ \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega + \mathrm{i}0} + \delta_{\mu-} \boldsymbol{e}_- \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega - \mathrm{i}0} - \boldsymbol{e}_\mu \right)$$

$$\times \left( \delta_{\nu+} \boldsymbol{e}_+^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega + \mathrm{i}0} + \delta_{\nu-} \boldsymbol{e}_-^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega - \mathrm{i}0} - \boldsymbol{e}_\nu^* \right) 2\pi\delta(\omega' - \omega). \tag{45}$$

Integrating over the frequency $\omega'$ of the scattered light, and dividing by the incident flux $F_0$ and the density of particles $N$, we arrive at the differential scattering cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2\hbar^2}{m^2c^2} \left( \boldsymbol{e}_-'^* \boldsymbol{e}_+ \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega + \mathrm{i}0} + \boldsymbol{e}_+'^* \boldsymbol{e}_- \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega - \mathrm{i}0} - \boldsymbol{e}_\mu'^* \boldsymbol{e}_\mu \right)$$

$$\times \left( \boldsymbol{e}_-' \boldsymbol{e}_+^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} - \omega + \mathrm{i}0} + \boldsymbol{e}_+' \boldsymbol{e}_-^* \frac{\omega_\mathrm{B}}{\omega_\mathrm{B} + \omega - \mathrm{i}0} - \boldsymbol{e}_\nu' \boldsymbol{e}_\nu^* \right). \tag{46}$$

Introducing the tensor

$$\Pi_{\mu\nu} = \delta_{\mu\nu} - \delta_{\mu+}\delta_{\nu-}\frac{\omega_B}{\omega_B+\omega+i0} - \delta_{\mu-}\delta_{\nu+}\frac{\omega_B}{\omega_B-\omega-i0}$$
$$= \begin{pmatrix} \omega/(\omega+\omega_B+i0) & 0 & 0 \\ 0 & \omega/(\omega-\omega_B+i0) & 0 \\ 0 & 0 & 1 \end{pmatrix}_{+,-,z} \tag{47}$$

the cross section can be written in a form

$$\frac{d\sigma}{d\Omega} = \alpha^2\frac{\hbar^2}{m^2c^2}|\langle e'|\Pi|e\rangle|^2. \tag{48}$$

This agrees with the result obtained by Ventura (1979). In contrast to the more conventional approach, the present formalism leads directly to the *square* of the matrix element for scattering. The second and third term in (34) can be interpreted as the interference terms between the 'seagull diagram' (contained in the first term) and the other diagrams (Kirk and Mészáros 1980).

It was easier to rederive the scattering cross section from the emissivity formula (6) but the opacity formula (5) could have been used as well. Then it would have been necessary to include the vacuum contributions in the photon field density matrix (15) that were omitted in the simplified correlation function (36). Since the dependence on the time branch dropped out of the velocity fluctuations (43) above, $\langle[\![u_\mu(\bar{t})u_\nu(\underline{0})]\!]\rangle = \langle[\![u_\mu(\underline{t})u_\nu(\bar{0})]\!]\rangle$, the commutator in the opacity formula will vanish. This is because of the simplifying assumptions that we have made. Stimulated scattering into the beam with flux $F_0$ will be exactly compensated by scattering from that beam.

## 6. Conclusions

We have seen that a line profile (for cyclotron absorption) and a continuum opacity (for scattering) can be derived from the same fundamental formula. It is reasonable to expect that, without the drastic simplifications made above, the Kubo formulae (5) and (6) will give accurate spectral lineshapes. They seem to be very general and provide a unified description of *all* radiation processes occurring in a plasma. Moreover, they appear to be the only way of arriving at meaningful expressions for opacity and emissivity. While the usual way of computing opacities as a sum of contributions due to different processes (absorption, scattering, two-photon processes, etc.) leads to useful approximations when one process is dominant, it leads to inconsistencies when several processes are considered together, as discussed in § 1. It is a mistake just to add the squares of Feynman diagrams for different processes—the Kubo formulae incorporate the interference terms between them in a natural way.

The distinction between absorption and scattering is, strictly speaking, unphysical. There is no way of distinguishing a scattering process from absorption followed by re-emission. The Kubo formulae make it superfluous to worry about this distinction because they contain all radiation processes (if we evaluate the current correlation functions with enough care). There is only one observable extinction coefficient, given by (5), and a separation of various radiation processes is meaningless. The Boltzmann equations, which depend on such a separation, should therefore not be applied to this problem.

The cross sections obtained in §§ 4 and 5 are, of course, not new results. What I hope to have shown is that the Kubo formulae are both fundamental and useful. The calculation of the scattering cross section in § 5 may appear unnecessarily complicated but the standard approach, if one uses the Stokes parameter formalism, also involves much algebra (see, e.g., Chou 1986). Our calculations were aided by the especially simple properties of a uniform magnetic field. But quantum field-theoretical methods must, of course, be applicable also to atoms. For realistic computations of differential scattering cross sections ('redistribution functions', e.g., Hubeny and Oxenius (1987)) further improvements of the calculational technique seem desirable. Basically, the method is straightforward, and perhaps not even difficult to implement in a computer code.

There is no answer to the question as to what is the correct value for the damping coefficient $\gamma$. Any $\gamma$ can only be approximate, because it corresponds to a *partial* summation of the perturbation series implied by the formal definition (8) of the current correlations. The value of $\gamma$ that leads to the best approximation for the opacities will in general depend on the radiation field. Thus the computation of opacities becomes coupled to the solution of the radiative transfer equation. This complication becomes even more severe when multiphoton processes are important, e.g., in model calculations for x-ray pulsars (Kirk *et al* 1986). In such problems it may prove to be easier to work directly with the correlation functions $\langle[\![A_\mu(x,t)A_\nu(0,0)]\!]\rangle$ and $\langle[\![j_\mu(x,t)j_\nu(0,0)]\!]\rangle$, rather than the more usual photon and electron distribution functions to which they are related. Of course, instead of the Boltzmann equations it is necessary to use more general transport equations, i.e. equations of motion for the correlation functions (Kadanoff and Baym 1962, Chou *et al* 1985). These equations will merit further investigation.

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